THE SIMPLE AND MODIFIED SIMPLE POISSON PROCESSES AND THE MAXIMUM-LIKELIHOOD ESTIMATORS OF THEIR PARAMETERS*

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SUMMARY

Tintner *et al.*[2] have considered the application of the simple and modified simple Poisson processes to the process of regional development. In their modified simple Poisson process the variable takes only discrete values $o, u, 2u, \ldots$ etc. In this note the joint probability-generating function of k equally spaced variables from the simple Poisson processes are obtained. Further the maximum-likelihood estimators of the parameters of the simple and the modified simple Poisson processes are obtained and the efficiencies of some simple least squares and generalized least squares estimators of the parameter of the simple Poisson process are worked out. As the maximum-likelihood estimator of u has a rather complicated form and a still more complicated variance, some simple asymptotically unbiased estimates of u on the lines of Tintner *et al.* and their variances are considered.

I. INTRODUCTION

Tintner *et al.* have considered the application of a modified version of the simple Poisson process in which the variable takes the values $o, u, 2u, \ldots$ etc. instead of $0, 1, 2, \ldots$, for the explanation of the trend of regional development. Both the simple Poisson process and the modified simple Poisson process lead to linear expected growth over time in the variable concerned. Tintner *et al*[1] use the least squares estimator to estimate the parameter of the simple

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Poisson process and then use the relationship between the variance and mean of the modified simple Poisson process to obtain an estimate of u.

With the simple Poisson process as the model the least squares estimate though unbiased is not an efficient estimator. In this note the maximum-likelihood estimate of the parameter of the simple Poisson process from equally spaced observations is worked out. It turns out to have a very simple form. For the modified version, however, the maximum-likelihood estimator has a very complicated form and its variance has a still more complicated form. Hence three estimates of u worked out on the basis of the method of moments are suggested and their asymptotic variances are compared. One of these is the same as the estimate of u suggested by Tintner *et al.*[1] except that the maximum-likelihood estimate of the parameter of lu where l is the parameter of the underlying simple Poisson process is used in place of the least squares estimate used by Tintner *et al.*[1] For the parameter of the simple Poisson process also, two other simple least squares and generalized least squares estimates are worked out and their efficiencies compared.

2. The Simple Poisson Process

A simple Poisson process is generated as follows. Let N_t be a discrete variable considered at time t, taking the values 0, 1, 2, ... etc. Let the probability that it changes from N to $N+1$ in the time-interval $(t, t+\triangle t)$ be the probabilities of all other transitions being of the order of $o(\triangle t)$. Further, let the changes in the time-interval $(t, t+\triangle t)$ be independent of changes in all the previous intervals.

Let

$P_N(\triangle t)$ be the probability that N_t has the value at time t.

Then

$$P_N(t+\triangle t)=P_{N-1}(t)\, l\triangle t+P_N(t)\,(1-l\triangle t) \quad N\geqslant 1$$

$$P_o(t+\triangle t)=P_o(t)\,(1-l\triangle t). \qquad \ldots(1)$$

Hence

$$P_N(t+\triangle t)-P_N(t)=P_{N-1}(t)\, l\triangle t-P_N(t)\, l\triangle t$$

and $P_o(t+\triangle t)-P_o(t)=-lP_o(t)\triangle t$.

Dividing both the sides by $\triangle t$ and taking limit as $\triangle t \to 0$ we get

$$\frac{\partial P_N(t)}{\partial t} = \left[P_{N-1}(t) - P_N(t) \right] l \qquad \ldots(1a)$$

and

$$\frac{\partial P_o}{\partial t} = -l P_o(t).$$

Multiplying the equations $(1a)$ by Z^N and adding over N from 0 to ∞ we get

$$\frac{\partial \pi}{\partial t} = l(z-1)\pi \qquad \ldots(2)$$

where π_t is the probability-generating function of N_t.

If $\qquad N = n_o$ at $t = 0$

then the solution of the equation (2) is given by

$$\pi = Z^{no} e^{l(z-1)t} \qquad \ldots(3)$$

If $\qquad n_o = 0$, this reduces to

$$\pi = e^{l(z-1)t} \qquad \ldots(4)$$

the solution considered in [1].

(4) gives the $p.g.f.$ of a Poisson distribution with mean given by lt. From the relationship (3) the conditional probability of N_t given n_o can be written down as

$$P(N_t = N/n_o) = \frac{e^{-lt}(lt)^{N-n_o}}{(N-n_o)!}. \qquad \ldots(5)$$

The process $\{N_t\}$ as defined above is a one-dependent Markoff process and hence using the formula (5) successively the joint probability of $N_1, \ldots, N_k$ given N_o can be written down. Thus

$$P(n_1, \ldots, n_k/n_o) = P(n_1/n_o)\, P(n_2/n_1) \ldots P(n_k/n_{k-1})$$

$$= \prod_{r=1}^{k} \frac{e^{-l}\, l^{n_r - n_{r-1}}}{(n_r - n_{r-1})!} \qquad \ldots(6)$$

Using the result (6) the joint $p.g.f.$ of $N_1 \ldots, N_k$ given n_o can be shown to be given by

$$\pi(z_1, \ldots, zk/n_o) = (z_1 z_2 \ldots z_k)^{n_o} e^{lz} \qquad \ldots(7)$$

where

$$Z = (z_k - 1) + (z_k z_{k-1} - 1) + (z_k z_{k-1} z_{k-2} - 1) + \ldots + (z_k z_{k-1} \ldots z_1 - 1).$$

For

$$\pi\,(z_1,\ldots,z_k/n_o)$$
$$=\sum_{n_1}\sum_{n_2}\ldots\sum_{n_k} P\,(n_1,\ldots,n_k/n_o)z_1^{n_1}\ldots z_k^{n_k}$$
$$=\sum_{n_1}\ldots\sum_{n_k}\prod_{r=1}^{k}\frac{e^{-l}\,l^{n_r-n_{r-1}}}{(n_r-n_{r-1})\,!}Z_r^{n_r} \qquad \ldots(8)$$

Summing first over n_k and using the result (3), (8) reduces to

$$\pi\,(z_1\ldots z_k/n_o)=\sum_{n_1}\ldots\sum_{n_{k-1}}\left[\prod_{r=1}^{k-1}\frac{l e^{-l}\,{}^{n_r-n_{r-1}}}{(n_r-n_{r-1})\,!}\right.$$
$$\left.\times\, z_1^{n_1}\ldots z_{k-2}^{n_{k-2}}(z_{k-1}\,z_k)^{n_{k-1}}\right]e^{l(z_{k-1})}$$

Summing over n_{k-1}, n_{k-2}, ... n_1 successively and making use of result (3) at every stage we arrive at result (7).

Hence the *c. g. f.* is given by

$$K\,(\theta_1,\theta_2,\ldots\theta_k/n_o)=n_o\,\Sigma\theta_r+l\,(e^{\theta_k}-1)$$
$$+\,l\,(e^{\theta_k+\theta_{k-1}}-1)+\ldots+l(e^{\theta_1+\cdots+\theta_k}-1). \qquad \ldots(9)$$

The means, variances and other moments can be obtained by expanding the exponential terms and simplifying them. In particular

$$E\,(N_r)=n_o+l_r$$
$$V\,(N_r)=l_r$$

and

$$\text{cov}\,(N_r,N_s)=l_s. \qquad r>s \qquad \ldots(9a)$$

Hence

$$\text{corr.}\,(N_s,N_r)=\sqrt{\frac{s}{r}} \qquad r>s.$$

The maximum-likelihood estimate of l can be obtained as follows.

The likelihood function of the observations n_1, ... n_k given n_o is given by (6) and its logarithm is given by

$$\log L=-lk+\Sigma\,(n_r-n_{r-1})\log l-\Sigma\log\,(n_r-n_{r-1})!$$

Differentiating log L with respect to l
we get

$$\frac{\partial \log L}{\partial l} = -k + \frac{n_k - n_o}{l}.$$

Hence the maximum-likelihood estimate of l is given by

$$\hat{l}_1 = \frac{n_k - n_o}{k} \qquad \text{...(10)}$$

$\hat{l}_1$ has surprisingly enough a very simple form, a form that is usually used by non-statisticians to estimate l, the coefficient of constant linear increase. The non-statisticians' estimate turns out to be the maximum-likelihood estimate if the observations came from a simple Poisson process and not from, as is usually assumed for the least squares estimate,

$$N_t = n_o + lt + \epsilon_t \qquad \text{...(11)}$$

where ϵ_t's have zero means identical variance $\sigma\epsilon^2$ and zero time-carrelation.

It can be verified that on the assumptions of the Simple Poisson Process model*.

$$V\left(\hat{l}_1\right) = \frac{l}{k}.$$

3. The Least Squares Estimate and its Efficiency for Models I and II.

The least squares estimate of l for given n_o is given by

$$\hat{l}_2 = \frac{\sum_{t=1}^{k} (n_t - n_o)\, t}{\sum_{t=1}^{k} t^2} \qquad \text{...(12)}$$

When the observations come from the process specified by (11) the least squares estimate (12) is the unbiased minimum-variance linear estimate of l. If further ϵ_t's are normally distributed, $\hat{l}_2$ is also the maximum-likelihood estimate of l. The variance of $\hat{l}_2$ under the assumptions (11) that is, for model II, is given by

$$V\left(\hat{l}_2\right) = \frac{\sigma\epsilon^2}{\sum_{t=1}^{k} t^2} = \frac{\sigma\epsilon^2\, 6}{k(k+1)(2k+1)}$$

* To be referred to as model I. The statistical model specified by equation (11) will be referred to as model II.

and that of $\hat{l}_2$ by $\sigma\epsilon^2|k^2$.

Hence the efficiency of $\hat{l}_1$ under the assumptions (11) is given by

$$\frac{V\left(\hat{l}_2\right)}{V\left(\hat{l}_1\right)}=\frac{6k}{(k+1)(2k+1)} \qquad \ldots (13)$$

$$\sim\frac{3}{k} \text{ for large } k.$$

$$\rightarrow 0 \text{ as } k\rightarrow\infty.$$

If, however, the observations come from a simple Poisson Process (model I)

$$V\left(\hat{l}_2\right)=\frac{l}{\frac{(2k+1)^2(k+1)^2}{36}}\left[\frac{(k+1)^2(2k+1)}{6}-R\right]$$

where

$$R=\sum_{r=1}^{k}\frac{r^4}{k^2}$$

$$=\frac{k+1}{k}\left[\frac{(k+2)(k+3)(k+4)}{5}-\frac{6(k+2)(k+3)}{4}+7.\frac{k+2}{3}-\tfrac{1}{2}\right]$$

and

$$V\left(\hat{l}_1\right)=\frac{l}{k}.$$

Hence

$$Eff.\left(\hat{l}_2\right)=\frac{V\left(\hat{l}_1\right)}{V\left(\hat{l}_2\right)}$$

$$=\frac{\frac{1}{k}}{\left[-\frac{R\cdot 36}{[(k+1)(2k+1)]^2}+\frac{6}{2k+1}\right]}$$

$$\rightarrow\frac{5}{6} \text{ as } k\rightarrow\infty \qquad \ldots (14)$$

when

$$k=2, \quad \frac{V\left(\hat{l}_1\right)}{V\left(\hat{l}_2\right)}=\frac{13}{25}$$

Hence as k increases from 2 to ∞ one would expect

$$\frac{V\left(\hat{l}_1\right)}{V\left(\hat{l}_2\right)}$$

to increase from $\frac{13}{25}$ to $\frac{5}{6}$. Hence for large k, $\hat{l}_2$ has an efficiency roundabout $\frac{5}{6}$ on the assumption of a simple Poisson Process whereas on the assumption (11), $\hat{l}_1$ has an efficiency tending to zero as $k \to \infty$.

Therefore unless the adequacy of the simple Poisson Process for the data is established as against that of the model (11), it may be better to use $\hat{l}_2$ rather than $\hat{l}_1$ particularly for large k.

It should be noted that $\hat{l}_1$ and $\hat{l}_2$ are unbiased for both the models.

4. Other Simple Least Squares and Generalized Least Squares estimates of l.

Just for the sake of comparison, a few more allied simple stochastic models for $\{N_i\}^a$ and the estimates based on them and the efficiencies of these and the previous estimates on the assumptions of the different stochastic models are considered below.

Since N_r has variance l_r on the assumption of the simple Poisson process, a comparable simple stochastic model is

$$\frac{N_r}{\sqrt{r}} = l\sqrt{r} + \epsilon_r \qquad \text{...(15)}$$

given $n_0=0$,

where ϵ_r's may be supposed to have zero means, identical variance $\sigma\epsilon^2$ and zero time-correlation. This statistical model will be referred to as model III.

The least-square estimate of l is then an unbiased minimum-variance linear estimate. It is obtained by minimizing

$$\sum_{r=1}^{k}\left(\frac{n_r}{\sqrt{r}}-\lambda\sqrt{r}\right)^2$$

and is given by

$$\hat{l}_3=\frac{\sum_{r=1}^{k} n_r}{\Sigma r}=\frac{2\sum_{r=1}^{k} n_r}{k(k+1)} \quad \ldots(16)$$ *

The variance of this estimate on the basis of model (15) is given by

$$V\left(\hat{l}_3\right)=\frac{2\sigma\epsilon^2}{k(k+1)}. \quad \ldots(17)$$

On the basis of the same model, *viz.*, model specified in (15) the variance of l_1 and l_2 are given by

$$V\left(\hat{l}_1\right)=\frac{\sigma\epsilon^2}{k} \quad \ldots(18)$$

and

$$V\left(\hat{l}_2\right)=\frac{9\sigma\epsilon^2}{(2k+1)^2}.$$

Hence

$$\frac{V\left(\hat{l}_3\right)}{V\left(\hat{l}_1\right)}=\frac{2}{k+1}\rightarrow 0 \text{ as } k\rightarrow\infty$$

and

$$\frac{V\left(\hat{l}_3\right)}{V\left(\hat{l}_2\right)}=\frac{2(2k+1)^2}{9k(k+1)}\rightarrow\frac{8}{9} \quad \ldots(19)$$

as $k\rightarrow\infty$.

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* It should be noted that in sections 3 and 4, the variances of the estimators $\hat{l}_1$, $\hat{l}_2$ and $\hat{l}_3$ of l for the three models I, II and III have been worked out separately and compared with each other. $\hat{l}_1$ $\hat{l}_2$ and $\hat{l}_3$ are the minimum-variance, unbiased, linear estimators BLUE of l models I, II, and III, espectively.

On the basis of the simple Poisson process

$$V\left(\hat{l}_3\right)=\frac{2}{3}\ \frac{(2k+1)}{k(k+1)}.\ l.$$

and

$$\frac{V\left(\hat{l}_1\right)}{V\left(\hat{l}_3\right)}=\frac{3k+1}{2(2k+1)}\rightarrow\frac{3}{4} \quad \text{... (20)}$$

as $k\rightarrow\infty$.

On the basis of the model given by (11)

$$V\left(\hat{l}_3\right)=\frac{4\sigma\epsilon^2}{k(k+1)^2}$$

and

$$\frac{V\left(\hat{l}_2\right)}{V\left(\hat{l}_3\right)}=\frac{3(k+1)}{2(2k+1)}\rightarrow\frac{3}{4} \quad \text{... (21)}$$

as $k\rightarrow\infty$.

Hence if only the three models, namely, the simple Poisson Process, the model given by (11) and the model given by (15) are considered as the possible alternatives $\hat{l}_2$ or the least squares estimate appears to be preferable either to $\hat{l}_3$ or to $\hat{l}_1$ particularly for large k since it has an efficiency roundabout 5/6 or more on the basis of any of the three models whereas $\hat{l}_3$ has an efficiency roundabout 3/4 or more and $\hat{l}_1$ has negligible efficiency on the basis of models other than the simple Poisson process.

If in (15) $\epsilon's$ are normally distributed $\hat{l}_3$ is known to be a maximum-likelihood estimate of l. However, even when $\epsilon's$ are not normally distributed but are distributed in a certain fashion to be described below, $\hat{l}_3$ turns out to be the maximum-likelihood estimate.

If $N_r's$ are distributed in a Poisson distribution with mean l_r as in the simple Poisson Process with $n_o=0$ but unlike as in the simple Poisson Process the $N_r's$ are distributed independently of each other, it can be shown that the maximum-likelihood estimate of l is given by $\hat{l}_3$.

For the likelihood function of $n_1, \ldots, n_k$ in this case is given by

$$L = \prod_{r=1}^{k} e^{-l_r} \frac{(l_r)^{n_r}}{n_r!}.$$

Hence $\log L = -l\Sigma r + \Sigma n_r(\log l + \log r) - \Sigma \log n_r!$

Differentiating with respect to l we get

$$\frac{\partial \log L}{\partial l} = -\Sigma r + \frac{\Sigma n_r}{l}.$$

Hence the maximum-likelihood estimate is given by

$$\frac{\Sigma n_r}{\Sigma r} \equiv \hat{l}_3.$$

This result raises the interesting question as to the forms of distribution of ϵ which render the least squares estimates identical with the maximum-likelihood estimates. This problem is, however, not tackled here.

Since the N_r's in the simple Poisson Process are also cross-correlated unlike as in the model given by (15) it may be interesting to work out the generalized least squares estimate when the ϵ_r's in (11) have the same co-variance matrix as N_r's except that the l in the co-variance martix may be replaced by an unrelated quantity σ^2.

The co-variance-matrix of the ϵ_r's, $r=1, \ldots, k$ is then given by

$$\Omega = \sigma^2 \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & 2 \ldots & 2 \\ 1 & 2 & 3 \ldots & 3 \\ 1 & 2 & 3 \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 2 & 3 & k \end{bmatrix}$$

It can be verified that Ω^{-1} is given by

$$\Omega^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 2 & -1 & & & & O \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ O & & & & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

Hence the generalized least squares estimate is obtained by minimizing

$$S^2 = (\underset{\sim}{n_r} - l_r)' \; \Omega^{-1} (\underset{\sim}{n_r} - l_r) \qquad \ldots(22)$$

w.r.t. l which can be shown to be given by $\frac{n_k}{k}=\hat{l}_1$ the maximum-likelihood estimate on the assumption of the simple Poisson Process. It should be noted that $\frac{n_k}{k}$ would be the maximum-likelihood estimate on the assumption of model (11) if ϵ_r's are normally distributed with the co-variance-martix given by Ω.

If σ^2 is replaced by l in Ω then the generalized least squares estimate can be shown to be given by

$$\hat{l}_4^2=\frac{2\sum_{r=1}^{k-1} n_r(n_r-n_{r+1})+n_k^2}{k} \qquad \ldots(23)$$

It can be verified that

$$E(\hat{l}_4^2)=l^2.$$

For large l the E's are normally distributed and $V(\hat{l}_4)$ can be shown to be asymptotically equal to $\frac{l}{k}$, same as $\hat{l}_1$.

5. The Modified Simple Poisson Process

The modified simple Poisson Process considered by the Tintner *et al.* is obtained by putting

$$y_t=N_t u.$$

Hence the c.g.f. of $y_1, \ldots y_k$ is given by

$$K(\theta_1 \ldots, \theta_k)=l\Big[\Big(e^{\theta_k u}-1\Big)+\Big(e^{(\theta_k+\theta_{k-1})u}-1\Big)+\Big(e^{(\theta_k+\ldots+\theta_1)u}-1\Big)\Big],$$

n_o and hence y_o being equal to 0.

All the moments can therefore be written down. In particular

$$E(y_r)=ul_r$$
$$V(y_r)=u^2 l_r$$

and $$\text{Cov}(y_r, y_s)=u^2 l_s \qquad r>s.$$

There are two parameters u and l to be estimated. For the maximum-likelihood equations, the likelihood given by

$$L=P(y_1, \ldots y_k/y_o=0)$$
$$=\prod_{r=1}^{k}\frac{e^{-l}\, l^{(y_r-y_{r-1})/u}}{\left(\frac{y_r-y_{r-1}}{u}\right)!}$$

has to be maximized w.r.t. l and u. The maximizing equations are given by

$$\frac{\partial \log L}{\partial l} = -k + \frac{y_k}{u\,l} = 0 \qquad \ldots(24)$$

and

$$\frac{\partial \log L}{\partial u} = -\log l \left(\frac{y_k}{u^2}\right) - \sum_{r=1}^{k} \frac{\partial \log \left(\frac{y_r - y_{r-1}}{u}\right)!}{\partial u} = 0$$

The first equation gives

$$u\hat{l} = \frac{y_k}{k},$$

which is similar to the estimate of l when $u=1$.

In the second equation the second term can be expressed as

$$\Sigma \left(\frac{y_r - y_{r-1}}{u^2}\right) \frac{\partial \log m!}{\partial m}$$

where

$$m = \frac{y_r - y_{r-1}}{u} \qquad r = 1, \ldots k$$

$$\frac{\partial \log m!}{\partial m} = \frac{\partial \log \int_0^\infty e^{-v} v^m dv}{\partial m}$$

$$= \frac{\int_0^\infty e^{-v} v^m \log v \, dv}{\int_0^\infty e^{-v} v^m dv} = \phi(m), \text{ say.}$$

Using the above result the second equation in (24) reduces to

$$\log l . \frac{y_k}{u^2} = \frac{\Sigma y_r - y_{r-1}}{u^2} \phi \left(\frac{y_r - y_{r-1}}{u}\right).$$

Cancelling u^2 in the denominator and putting $\log l = \log lu - \log u$ and simplifying we get

$$\log u = \log \hat{lu} - \frac{\Sigma z_r \phi(z_r)}{\Sigma z_r} \qquad \ldots(25)$$

where

$$z_r = \frac{y_r - y_{r-1}}{u}.$$

This equation can be solved by the method of iteration with a trial value of u to start with.

It is difficult to obtain an explicit expression for the variance of $\hat{l}$ and $\hat{u}$. The covariance matrix of $\hat{l}$ and $\hat{u}$ for large k is given by

$$\begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial l^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial l\,\partial u}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial l\,\partial u}\right) & -E\left(\frac{\partial^2 \log L}{\partial u^2}\right) \end{bmatrix}^{-1}$$

with $$-E\left(\frac{\partial^2 \log L}{\partial l^2}\right)=\frac{k}{l}$$

$$-E\left(\frac{\partial^2 \log L}{\partial l\,\partial u}\right)=\frac{k}{u}$$

and $$-E\left(\frac{\partial^2 \log L}{\partial u^2}\right)=-2\log l.\frac{lk}{u^2}$$

$$+\sum_{r-1}^{k} E\left\{\frac{\partial^2 \log\left(\frac{y_r-y_{r-1}}{u}\right)!}{\partial u^2}\right\}$$

The last term in $E\left(\frac{\partial^2 \log L}{\partial u^2}\right)$ is difficult to evaluate.

6. Some Simple Estimates of u.

Simpler but less efficient estimates of u like that of u in [1], whose variances can be more easily worked out than that of the maximum-likelihood estimator, can be obtained. Three such simple estimates are listed below:

$$\hat{u}_1=\frac{\Sigma(y_r-\hat{ul}_r)^2}{\Sigma_r\left(1-\frac{r}{k}\right)\hat{ul}} \qquad ...(26)$$

where $\hat{ul}$ is the maximum-likelihood estimator of ul, given by (25). It can be shown that

$$E(\hat{u}_1) \sim \frac{E\Sigma(y_r-\hat{ul}_r{}^2)}{\Sigma r(1-r/k)}\Big/E(\hat{ul})$$

$$=u.$$

The estimate considered by Tintner[1] is similar to $\hat{u}_1$ and is given by

$$\hat{u}_1' = \frac{\Sigma(y_r - \hat{\hat{u l}} r)^2}{\Sigma r . \hat{\hat{u l}}}, \qquad \ldots(27)$$

where $\hat{\hat{ul}}$ is the least squares estimate given by

$$\hat{\hat{ul}} = \frac{\sum_{t=1}^{k} t y_t}{\sum_{t=1}^{k} t^2}.$$

Since $\hat{ul}$ is a maximum likelihood estimate of ul, $\hat{u}_1$ may be a more efficient estimate than $\hat{u}_1'$ when $\hat{u}_1'$ is corrected for its bias. To correct for the bias in $\hat{u}_1'$ which does not vanish even for large k, we first obtain $E(\hat{u}_1')$.

$$E(\hat{u}_1') \sim \frac{E(\Sigma(y_r - \hat{\hat{ul}} r)^2/\Sigma r,}{E(\hat{\hat{ul}})}$$

$$E(\hat{\hat{ul}}) = ul$$

and it can be shown that

$$E\left(\frac{\Sigma(y_r - \hat{\hat{ul}} r)^2}{\Sigma r}\right) = u^2 l \left(\frac{\Sigma r^4 - \Sigma r^2 . \Sigma r}{\Sigma r^2 . \Sigma r}\right)$$

$$= u^2 l \left[\frac{1}{5} + O\left(\frac{1}{k}\right)\right].$$

Hence

$$E\left(\frac{\Sigma(y_r - \hat{\hat{ul}} r)^2 \Sigma r^2}{\Sigma r^4 - \Sigma r^2 . \Sigma r}\right) = u^2 l.$$

Therefore

$$E\left\{\frac{\Sigma(y_r-\hat{\hat{u}}lr)^2.\Sigma r}{(\Sigma r^4-\Sigma r^2.\Sigma r)\hat{\hat{u}}l}\right\}=u+O\left(\frac{1}{k}\right)$$

Hence
$$\frac{\Sigma(y_r-\hat{\hat{u}}lr)^2\Sigma r^2}{(\Sigma r^4-\Sigma r^2.\Sigma r)\,\hat{\hat{u}}l}$$

is an unbiased estimate of l for large k.

Another estimate similar to $\hat{u}_1$ is given by

$$\hat{u}_2=\frac{\dfrac{1}{k}\sum_{r=1}^{k}\dfrac{(y_r-\hat{u}lr)^2}{r(1-r/k)}}{\hat{u}l} \qquad \text{...(28)}$$

It can be verified that

$$E(\hat{u}_2)=u+O\left(\frac{1}{k}\right).$$

One would expect $\hat{u}_1$ to be a more efficient estimate than $\hat{u}_2$ but this has to be investigated. A third estimate of the type which is usually used when the observations are correlated is

$$\hat{u}_3=\sum_{r=1}^{k}\frac{(y_r-y_{r-1}-\hat{u}l)^2}{(k-1)\hat{u}l} \qquad \text{...(29)}$$

It can be verified that

$$E(\hat{u}_3)=u+O\left(\frac{1}{k}\right).$$

Actually

$$E\left\{\sum_{r=1}^{k}(y_r-y_{r-1}-\hat{u}l)^2\right\}=(k-1)u^2l.$$

One would expect that at least for small k, $\hat{u}_3$ would have a smaller variance than either $\hat{u}_1$ or $\hat{u}_2$.

For working out the variances of $\hat{u}$ the following formula is used.

$$V(\hat{u})=\frac{V(\text{num})}{[E(\text{denom})]^2}-2\text{cov}\frac{(\text{num., denom.})E(\text{num.})}{[E(\text{denom.})]^3}+\frac{V(\text{denom.})[E(\text{num.})]^2}{[E(\text{denom.})]^4}+O\left(\frac{1}{k^{3/2}}\right). \quad ...(30)$$

Making use of this formula and noting that for $\hat{u}_3$

$$E(\text{num.})=u^2l$$

$$E(\text{denom.})=ul$$

$$V(\text{num.})=\frac{u^4l}{k^2}+\frac{2u^4l^2}{k-1}$$

and $$\text{cov}(\text{num., denom.})=\frac{u^3l}{k}$$

$V(\hat{u}_3)$ can be simplified and it reduces to the expression

$$V(\hat{u}_3)=\frac{2u^2}{k-1}+O\left(\frac{1}{k^{3/2}}\right) \quad ...(31)$$

Similarly $V(\hat{u}_1)$ can be worked out and simplified. It reduces to

$$V(\hat{u}_1)=\frac{4u^2}{5}+\frac{u^2}{5lk}+O\left(\frac{1}{k^{3/2}}\right) \quad ...(32)$$

Hence for sufficiently large k at least

$$V(\hat{u}_3) < V(\hat{u}_1)$$

and hence the estimate $\hat{u}_3$ is more reliable than $\hat{u}_1$ at least for large k.

It is theoretically possible to work out the variance of $\hat{u}_2$ and compare it with those of $\hat{u}_1$ and $\hat{u}_3$ but because of the form of $\hat{u}_2$, reduction of the expression for the variance is very much more complicated than in the case of $\hat{u}_1$ and $\hat{u}_3$ and it has not been attempted here. A capital intensive method (Monte Caulo method) of working out the variances for different sets of numerical values of the parameters involved and of k is possible but firstly it does not provide a comprehensive answer and secondly at this stage at least it does not seem to be worth the trouble.

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